Leone Learning Systems, Inc.
Wonder. Create. Grow.

Leone Learning Systems, Inc.
237 Custer Ave
Evanston, IL 60202
Email tj@leonelearningsystems.com
Web site: www.leonelearningsystems.com

Phone 847 951 0127
Fax 847 733 8812

Turtle Geometry

For primary and elementary classrooms

TJ Leone

June 2004

Introduction

This article describes a number of exercises in Turtle Geometry, a computational (though not necessarily computer-based) style of geometry that provides a complementary perspective to Euclidean geometry as presented in the Montessori classroom. The exercises reflect my own personal experience and understanding of the practice and philosophy of Montessorians and those of another constructivist community, *constructionists*.

Constructionists share with Montessorians the constructivist belief that ideas are constructed in the mind of the learner, rather than being transmitted from teacher to learner. Seymour Papert, the founder of constructionism, was a student of Piaget's, and the name "constructionist" was chosen in part because of its similarity to the word constructivist.

What distinguishes constructionists from other constructivists is the idea that this construction of knowledge, which "takes place 'in the head' often happens especially felicitously when it is supported by construction of a more public sort 'in the world'—a sand castle or a cake, a Lego house or a corporation, a computer program, a poem, or a theory of the universe (Papert, 1993a)."

On the other hand, for Montessorians, "children construct their own knowledge through exploration and discovery assisted by the introduction of efficient experiences planned by the teacher (Loeffler, 1992)." Because Montessorians are constructivists, the term "efficient experiences" is not used to refer to efficiency in transferring knowledge or "training" the child in any way. Rather, the term "efficient experiences" as applied to the child may be compared to experiences that would efficiently support scientific discovery[1].

[1] For example, when conducting an experiment, the scientists generally try to reduce the number of variables as much as possible. A corresponding behavior in the Montessori classroom is the isolation of difficulty, in which Montessorians try to make aspects of the world apparent by providing opportunities to compare, contrast, serialize objects and otherwise manipulate objects that are identical except for the attributes to be studied.

It must also be noted that the most efficient route to a scientific discovery is never a straight line. Scientists must spend considerable time repeating experiments, examining results, and "mulling things over" in a process that is largely subconscious. When Montessorians speak of "efficient experiences", they refer to materials and exercises that support these processes in an efficient way. Montessorians do not look for "shortcuts" around these processes which are critical for the construction of knowledge.

Introduction (continued)

Neither the Montessori nor the constructionist community sees “construction in the world” and “efficient experiences” as mutually exclusive paths to the construction of knowledge. Constructionists generally acknowledge that scaffolding provided by the teacher and the environment are important for successful construction and reflection on construction. On the other hand, Montessori children are typically free to explore materials in ways that were not presented by their teacher (Chattin-McNichols, 1992) and Montessorians recognize the importance of supporting open-ended, creative tasks, especially for children older than five.

It is my conviction that the Montessori and constructionist communities have much to offer each other. In this article, I will present a set of exercises based on Montessori’s work that help prepare the child for engaging in work that is commonly done in constructionist classrooms. These exercises were selected to highlight a particular way in which the Montessori and constructionist approaches can be mutually beneficial. There are certainly many others.

Logo

Although constructionism embraces more than just a style of computer-based learning, Seymour Papert is best known as one of the originators of Logo, a computer programming language developed for children. Logo typically provides onscreen objects, called Turtles, which have at least two attributes—heading and position. Children can cause Turtles to traverse various paths by giving them commands such as FORWARD 50, which tells the Turtle "Change position by moving 50 pixels in the direction of your current heading", or RIGHT 90, which tells the Turtle "Change your heading by pivoting in place by 90 degrees to the right".

Children can also tell the Turtle to "put down" a pen which causes it to leave a trail as it moves. In this way, children can use the program to draw various shapes. The work required to create particular shapes or analyze the shapes created by particular commands forms the basis of Turtle Geometry.

In his 1985 book, *The Computer and the Child: A Montessori Approach*, Peter Gebhardt-Seele (Gebhardt-Seele Peter, 1985) suggests that Turtle Geometry, as it is implemented in the Logo programming language, might be a useful addition to the Montessori classroom.

However, at the time Gebhardt-Seele's book was written, "very little [had] been actually proposed [by the Logo community] in ways that [could] be implemented in the classroom (Hoyles & Noss, 1992)." Gebhardt-Seele proposed Logo exercises that might be appropriate for the Montessori classroom, including exercises with Turtle Geometry, with the caveat that more study of such exercises was needed.

Logo in the United States, after an initial period of wide popularity, great enthusiasm, and high expectations, succumbed to attacks on two fronts—those who felt that computers (or computer applications designed for pedagogical reasons) were inappropriate for the classroom, and those who felt that Logo was "outdated" (after all, it was invented way back in the sixties).

In recent years, "there has been a flurry of new Logo development accompanied by renewed public awareness and enthusiasm (*What is Logo?*, 2000)." Variants of Logo have been developed to incorporate more recent developments in programming such as multi-media development and object-oriented programming as well as massively parallel versions of Logo. Important innovations have also been made in the development of "digital manipulatives," LEGO Logo products that children can use to build and program robots[2].

[2] The trajectory of Logo's popularity may sound familiar to students of the history of the Montessori movement.

Distinguishing Logo and Turtle Geometry

Although Turtle Geometry grew out of work on Logo, it is important to recognize them as two distinct constructs before we consider a Montessori approach to Turtle Geometry. Turtle Geometry is a style of geometry, just as Euclidean geometry and Cartesian geometry are styles of geometry (Papert, 1993b). Logo is a programming language.

As a programming language, Logo supports much more than just exercises in geometry. It is a derivative of Lisp, a powerful list processing language that is still used today as a teaching tool for computer science students and as a research and development tool, most notably in the area of artificial intelligence. Like Lisp, all versions of Logo have powerful list processing capabilities. The richness of the language might be captured by reference to Brian Harvey's three volume text *Computer Science Logo Style*. Of the fifteen chapters in the first volume, only one chapter is devoted to Turtle Geometry. There is virtually no discussion of Turtle Geometry in the other two volumes (Harvey, 1997).

As mentioned in the previous section, a broad range of development and learning environments for Logo have been created, with a wide range of features beyond Turtle Geometry and list processing. These environments support a range of activities that include game programming, creating multi-media storybooks or reports, or programming robots.

Turtle Geometry, on the other hand, need not be confined to the Logo environment. For example, there are implementations of Turtle Geometry in other computer languages. More importantly for this discussion, there are off-computer activities and experiences that are crucial for an understanding of Turtle Geometry. There are also computer-based activities that don't involve programming, but can still help children construct insights into Turtle Geometry, and link it to other work they do in the Montessori classroom.

Many people who work with Logo mention offline exercises to help develop children's awareness of shapes (such as finding rectangles in a room, or making one out of clay) before they try to construct one on the screen. During Logo activities that involve Turtle Geometry, a key off-line activity is "playing Turtle". The child takes on the role of the screen Turtle in order to anticipate or evaluate the path taken by the Turtle. All of these activities are considered as Turtle Geometry (or related) activities.

Comparing Montessori's and Papert's Approaches to Geometry

In devising her approach to geometry, Montessori took note of the fact that children are aware of objects as a whole before they grasp components like sides and vertices or measurements such as length, area, and angle. She then proceeded to develop materials to give them "efficient experiences" with shapes and gradually guide them toward analyses of different geometric objects in preparation for work in formal Euclidean geometry.

Seymour Papert's work on Turtle Geometry is based on the child's awareness of her own body and other mobile bodies such as dogs, cats, and boats. Living things and motorized vehicles can change their positions on a surface by changing their heading (turning right or left some number of degrees) and then moving forward or back by some distance.

Turtle Geometry as implemented in Logo gives children the opportunity to construct computer programs and screen images "in the world" to which children could relate. The movements of the Turtle resonates with the movement of the child's body, and the requirements of creating Turtle Geometry objects make mathematics meaningful, as opposed to the dissociated learning of mathematics that takes place in traditional classrooms. Papert compares the rote learning of mathematics to a "dance class without music or dance floor" (Papert, 1993b).

Benefits of Turtle Geometry for Montessorians

When used with a carefully considered Montessori approach, Turtle Geometry can provide important support for children's work in geometry.

Since Turtle Geometry involves the analysis of trajectories, children who use it come to see shapes as created by actions. For example, they might describe a square as a shape that is made by "walking forward some amount, turning right 90 degrees, then doing the same thing three more times". This gives them an alternative perspective on the square, on sides of a figure, on angles, and on external angles. If proper connections are eventually made between this perspective and corresponding Montessori materials, the child's concept of the shapes are strengthened.

"Playing Turtle" provides a perspective on geometry that might be especially useful to learners who favor what Gardner calls the "bodily/kinesthetic" style of learning.

In classrooms where Logo software is available on computers, Turtle Geometry figures in Logo can only be constructed by reference to measurements of lengths and angles. When children are ready, this can be a useful lever to help them progress to higher levels of geometric thought.

Further, children can create a much wider range of shapes with Logo than they can with Montessori materials. These include recursive designs that are found in nature (spirals and snowflakes, for example) and rich space-filling designs that could be used to revisit the exercises in design with metal insets at a higher level.

Why I Mix Turtle Geometry and Montessori

Papert felt that by creating an environment where mathematical analysis is a valuable tool that can help produce meaningful results, children would spontaneously develop capabilities in mathematical analysis. The environment he created together with Bobrow and Feurzeig, permits deep explorations of a broad range of mathematical (and linguistic and scientific) ideas for learners from elementary school through graduate school and beyond.

Still, Turtle Geometry as implemented in Logo is not without its problems. Reports of children spontaneously performing mathematical analysis with Logo are rare (Hoyles & Noss, 1992). Children tend to work with Logo at the perceptual level, tweaking their programs by trial and error until the desired effect is produced. In a literature review on the effects and efficacy of the Logo learning environment, it was found that "without guidance, misconceptions [in geometry] can persist (Clements & Meredith, 1992)." Further, "some studies show limited *transfer* to activities outside of Logo (Clements & Meredith, 1992)."

Clements and Meredith further reported:

"Exposure [to Logo] alone is not completely adequate. A more satisfactory approach features *teacher mediation* and a sound theoretical foundation (e.g., for geometry: Piaget and van Hiele). Mediation implies clarification of the mathematics in Logo work and the extension of the ideas encountered; construction of links between Logo and non-Logo work; and provision of some structure for Logo tasks and explorations. Structure does not imply authoritarianism. For example, it is often useful to allow hesitant students to accept or reject suggestions until they build confidence."

"Construction of links between Logo and other mathematics activities [is] challenging--research shows that teachers find it extremely difficult to create a learning environment that fosters creativity within [traditional] school and curricular structures." Further, children must typically be exposed to Logo for more than one year before any noticeable effects can be observed (Clements & Meredith, 1992). This is less likely to happen in traditional schools where children spend only one year with a particular teacher and there are seldom school-wide policies about the kind of software to be used or how its use should be integrated into other classroom activities.

As I read about (and experienced) this problem reported by Clements and Meredith, it occurred to me that a Montessori approach could provide the structure that is so often lacking in Logo activities. In the next section, I will describe a set of Montessorian "efficient experiences" that I developed to support children's work in Turtle Geometry. These activities provide "links between Logo and non-Logo work [as well as] provision of… structure for Logo tasks and explorations" (Clements & Meredith, 1992) using a Montessori approach.

Why I Mix Turtle Geometry and Montessori (continued)

The exercises presented below are not intended to replace geometry exercises or projects in Montessori or constructionist classrooms. Rather, the intent is to provide supplementary exercises that can be used in either setting. I have tried out most of the exercises as part of an enrichment program at Northwestern University's Center for Talent Development. Where preliminary examinations of process and outcomes have been conducted, they have shown quite promising results.

All of the exercises below are intended to be preliminary to children's exploration of Turtle Geometry with Logo.

Off-line Turtle Geometry exercises for the Montessori classroom

One of the difficulties in Turtle Geometry is relating the "bird's eye view" one gets when looking at the screen to the perspective of the Turtle. For example, a right turn is always a turn in the clockwise direction. For children who know their right from their left, it is not a difficult task to pivot in place to the right. In this case, they are "playing Turtle", executing a Turtle command by taking the perspective of the Turtle.

However, turning the Turtle on the screen is another matter. Suppose that the Turtle has been given commands that cause it to have a heading of 180 degrees. On the screen, it will appear to be facing downward. Many children have trouble understanding why this Turtle ends up facing "left" (heading of 270 degrees) after being given the command to turn right (clockwise) by 90 degrees (Figure 1).

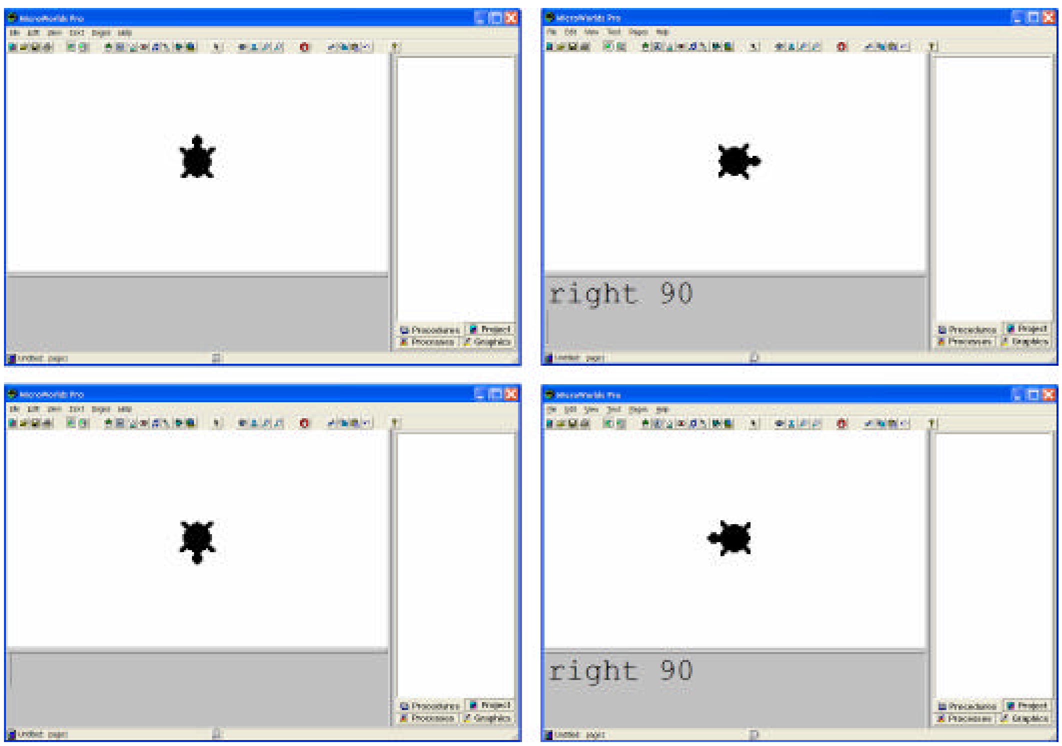

Figure 1. When the turtle begins with a heading of zero, the equivalent of a compass heading of North on most maps (upper left panel), children typically have little difficulty anticipating that a turn of ninety degrees to the right will change the turtle's heading to 90 degrees or East (upper right panel). However, when the turtle begins with a heading of 180 degrees, the equivalent of a compass heading of South, children are often surprised that the resulting heading is 270 degrees, the equivalent of a compass reading of West. Children are confused because they orient themselves according to the left and right sides of the screen rather than orienting themselves relative to their last position. From this perspective, the turtle appears to be facing left after a turn of RIGHT 90, which seems a contradiction. It is only with experience that they come to recognize that a right turn is a turn in the clockwise direction relative to the turtle's last heading. Thus, RIGHT 90 changes a heading of 0 to a heading of 0 + 90 = 90, and changes a heading of 180 to a heading of 180 + 90 = 270.

Off-line Turtle Geometry exercises for the Montessori classroom

Without force feeding the connection to the child, all Turtle Geometry exercises should have two versions: one with the Turtle’s perspective, and one with the bird’s eye view. The child should have ample opportunity to spontaneously develop awareness of the connection between these perspectives.

Extensions to exercises in walking the line and tracing shapes

In the primary classroom, walking the line and tracing geometric insets present an opportunity to begin making connections between the Turtle perspective and the bird's eye view. The teacher should provide opportunities for children to walk lines that correspond to various shapes in the geometric cabinet, and, when the child is ready, to compare the shape made by the line to the shapes in the cabinet.

From about the age of six, Montessori began work on geometric analysis informally:

"The geometric analysis of figures is not adapted to very young children. I have tried a means for the *introduction* of such analysis, limiting this work to the *rectangle* and making use of a game which includes the analysis without fixing the attention of the child upon it. This game presents the concept most clearly.

The *rectangle* of which I make use is the plane of one of the children's tables, and the game consists in laying the table for a meal. I have in each of the "Children's Houses" a collection of toy table-furnishings, such as may be found in any toy-store. Among these are dinner-plates, soup-plates, soup-tureen, saltcellars, glasses, decanters, little knives, forks, spoons, etc. I have them lay the table for six, putting *two places* on each of the longer sides, and one place on each of the shorter sides. One of the children takes the objects and places them as I indicate. I tell him to place the soup-tureen in the *centre* of the table; this napkin in a *corner*. 'Place this plate in the centre of the short *side*.'
Then I have the child look at the table, and I say, 'Something is lacking in this *corner*. We want another glass on this *side*. Now let us see if we have everything properly placed on the two longer sides. Is everything ready on the two shorter sides? Is there anything lacking in the four corners (Montessori, 1964)?'"

For Turtle Geometry, informal analysis might begin during exercises in walking the line. "Go forward until you get to the corner and stop. Turn right. Go forward until you get to the next corner. Turn right." Commands can be used to synchronize action for four children walking a rectangle. Children also enjoy giving commands to the teacher.

This informal analysis helps children work through another problem they encounter when they first "play Turtle". Children typically have trouble distinguishing change in position from change in direction. In common experience, when we hear or give the command "turn left", we understand this to mean "change your heading to more or less 90 degrees to the left and continue moving forward". From the Montessori perspective, it is important for children to understand the difference between position and direction, just as it is important for children to learn, for example, the difference between thickness and height (through activities with the knobless cylinders and other materials).

In later activities, after the child has begun counting sides and angles of shapes from the geometric cabinet, the child may be asked to count the number of tiles they traverse (or steps they take) when walking a line from one corner to the next. This is another part of analysis of change in location—measuring the amount of change. Later exercises give the child the opportunity to measure change in heading.

For the above exercises in walking the line, the child is taking the Turtle's point of view. A bird's eye version of the activity should be done after the child has done the first line walking activity above and has begun matching geometric insets to outline cards. In this activity, the child uses a toy Turtle to walk around the outline of a shape, for example, a rectangle. The teacher should introduce this exercise by saying, "This is a turtle. The turtle is going to walk the line of this rectangle."

The teacher should be very deliberate about separating change in position from change in location, i.e., slide the turtle from one corner to the next without changing its direction and stop. When the turtle reaches the corner, pivot in place without changing position until the turtle had turned to the direction of the next side, at which point the teacher again deliberately pauses. Next, the teacher slides to the next corner without changing direction, etc. The teacher may wish to substitute an animal from the farm material used in the language section or some other appropriate object for the turtle.

After the child has experience with the exercise, the teacher should see if the child spontaneously counts changes in position and changes in direction. If not, the teacher may introduce this as an activity, counting changes in position (sides of the figure) in one presentation and changes in position (turtle turns) in another.

An exercise to correspond to the counting of tiles while walking the line could be to place rectangles on graph paper and count squares as the child moves a toy Turtle from one corner to the next. The graph paper should be drawn so that the side of each rectangle in the geometric cabinet can be measured in some whole number of squares. This exercise can also be used as preparation for measuring with a ruler (In *The Computer and the Child: A Montessori Approach*, Peter Gebhardt-Seele suggests a similar activity in which the child uses a toy Turtle with a pen to trace figures on graph paper).

Colored Sectors

Turtle Geometry has weaknesses in its presentation of the angle concept which I will discuss later. At this point, I would like to address a problem I have with the Montessori approach to angles that impacts understanding of both Euclidean Geometry and Turtle Geometry. Studies have shown that when they are first learning about angles, children typically confound angle size with the length of the sides in the representation of the angle, or the area between those sides (Prescott, Mitchelmore, & White, 2002).

I love the fraction circles, and I use them regularly, along with the instrument for measuring the angles of fractions. However, they don't address the problem with angle measurement mentioned above. If anything, given experience measuring lines with rulers, it probably looks to most children as if the instrument for measuring angles of fractions is actually measuring the length of the fraction's arc. This problem is mitigated in later exercises with geometry sticks, but we can use the Montessori approach to support better angle concept development in both Euclidean Geometry and Turtle Geometry.

When Montessori wanted to help the child distinguish the dimensions of thickness and height, she created four sets of cylinders that varied as follows: (1) height increases as thickness increases, (2) height is constant as thickness increases, (3) thickness is constant as height increases, and (4) height decreases as thickness increases.

A similar approach can be taken to help the child distinguish the dimensions of a sector of a circle with four sets of sectors (Figure 2).

Colored Sectors (continued)

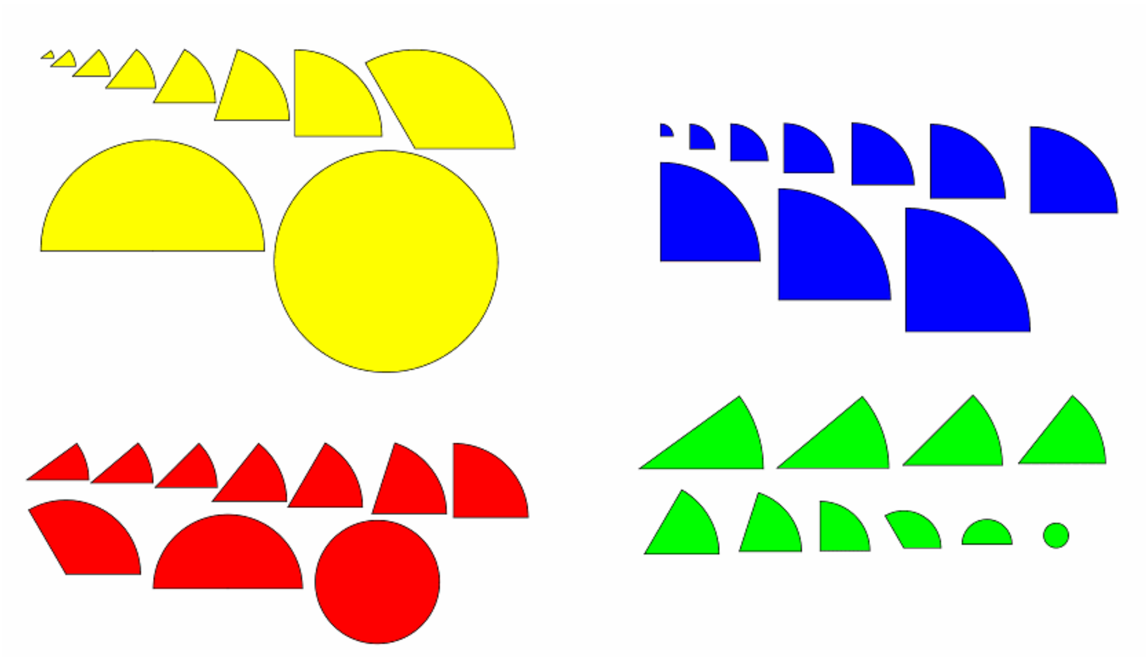

Figure 2. The yellow sectors can be placed in increasing order by both radius length and angle. The red sectors can be ordered by angle. The blue sectors can be ordered by radius length. The green sectors can be placed in increasing order by angle and simultaneously in decreasing order by radius.

Exercises with these sectors should precede measurement of angles with the instrument for measurement of angles. When measurement of angles is shown, the child should have the opportunity to compare the red fraction with yellow and green sectors that have the same angle measurement. In preparing these sectors, it is important to make them large enough so that children can make the necessary distinctions, especially with the green sectors.

Matching Polygons and Fraction Circles

After the child has completed the activity of counting sides and angles of polygons and has been presented the fraction circles, she can be shown how to line up fraction circle sectors with the external angles of polygons (Figure 3). This prepares the child for the Total Turtle Trip Theorem with regular polygons. Later exercises should have the child "walk" the outline of the polygon with a Turtle. This exercise (and web-based variations) has proven effective in tutoring sessions and class work. The relative sizes of the fractions and polygons in the picture below do not reflect the relative sizes of Montessori fraction circles and polygons from the geometric cabinet. However, the same angle relations apply.

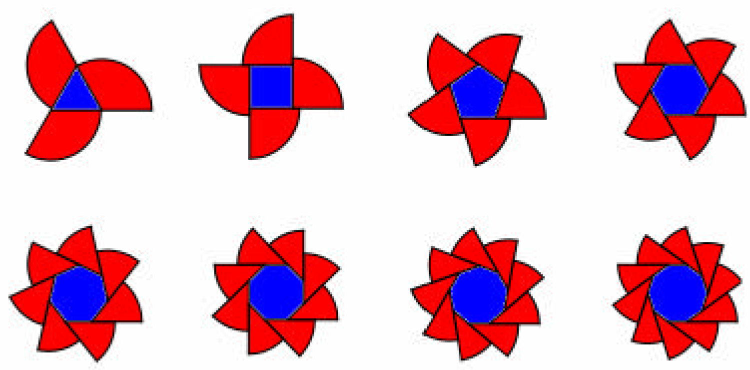

Figure 3. Polygons and the sectors of a fraction circle can be arranged to show that the external angles of regular polygons add up to 360 degrees.

This exercise has a number of important points of interest which may be highlighted in separated presentations. One of these is the correlation between Turtle turns and the red sectors. To trace the polygon, the Turtle moves forward along the side of the polygon until it reaches a sector (fraction). At this point, its heading is in line with one of the radii of the sector. The Turtle then turns until its heading is in line with the other radius of the sector. This is the correspondence between Turtle turn and angle—turning from one ray of the angle (radius of the sector) to the other.

Matching Polygons and Fraction Circles (continued)

Another point of interest is that the internal (blue) angles of the polygon are supplementary (form a straight line or straight angle with) the external (red) angles. Relating this to Turtle turns, we can also see that, in general, the angle drawn by a Turtle is supplementary to the Turtle turn used.

Further, we can note that the external angles of all of the polygons add up to 360 degrees. This is clear because every piece in the *n*-piece fraction circle is used to form the external angles of the *n*-sided polygon.

This leads us to see that each external angle of a regular *n*-gon is 360 / *n* degrees. Eventually, we can also help the child recognize that, since the internal angles are supplementary to the external ones, the internal angles of an n-gon is 180 – 360 / *n* degrees, or 180 (*n*-2) / *n* degrees.

Of course, although I use the variable *n* in discussing these presentations, the Montessori approach of moving from arithmetic to algebra should be used, and exercises in measurement and algebra come much later than the initial presentation.

Clock Work

Here is another activity for distinguishing change in direction from change in position which also gives experience in assigning numbers to headings and turns. A circle is laid out on the floor with tape. Straight pieces of tape divide the circle into twelve equal pieces. At each intersection of straight line and circle, the child places numbers in order from one to twelve. The numbers could be placed in a standing picture frame so that a child standing in the middle of the circle can see each number clearly.

The child in the center of the circle plays the role of an hour hand. A first exercise might simply be to arrange the numbers correctly. Children can also "play clock" in pairs or with a teacher, following commands to "face 4", "face 9", etc.

In later exercises, children make turns relative to current position. For example, suppose that the child is facing the number 5. If the command is "turn the right way 3 hours", the child turns to face the number 8. If the child is facing a 7 and the command is "turn the wrong way 6 hours", the child turns to face the 1.

Eventually, this game may be used to solve math problems. For example, to solve the expression 9 + 2, the child faces 9 and then turns "the right way" 2 hours to face 11. Subtraction is done by facing "the wrong way". Gradually, more and more complicated expressions can be solved this way, such as 3 + 7 – 4 – 2 + 6.

A special point of interest is that, with the circle, it is possible to solve problems like 1 – 5 or 9 + 9 = 6. Before the child has done long division, it is enough simply to make note of this odd phenomenon, which lays the groundwork for modular arithmetic. After the child has done long division, the clock material could be used to show remainders of division by 12. This activity is also preparation for understanding angles greater than 360 degrees and less than zero degrees.

To further distinguish change in direction from change in position, commands may be added to walk forward to the number from the center of the circle, and to walk backward from the number to the center.

For the "bird's eye" version of this exercise, a toy clock can be used.

The expression "the right way" helps the child associate the direction "right" with the clockwise direction, and eventually to associate "left" with the counter-clockwise direction. Some children may find this exercise useful in learning to distinguish right from left. Further, associating "right" with "clockwise" can help the child relate the Turtle perspective (standing in the middle of a circle) with the bird's eye view (looking down at a clock).

•
•
•
•
•
•
•
•

Further extensions of walking the line and tracing shapes

After this exercise, the child may revisit walking the line, this time with angle measurement. At each corner of a figure on the floor, there is an arrangement of numbers corresponding to the twelve hours of the clock. At each corner, the child counts how many "hours" she turns (Figure 4).

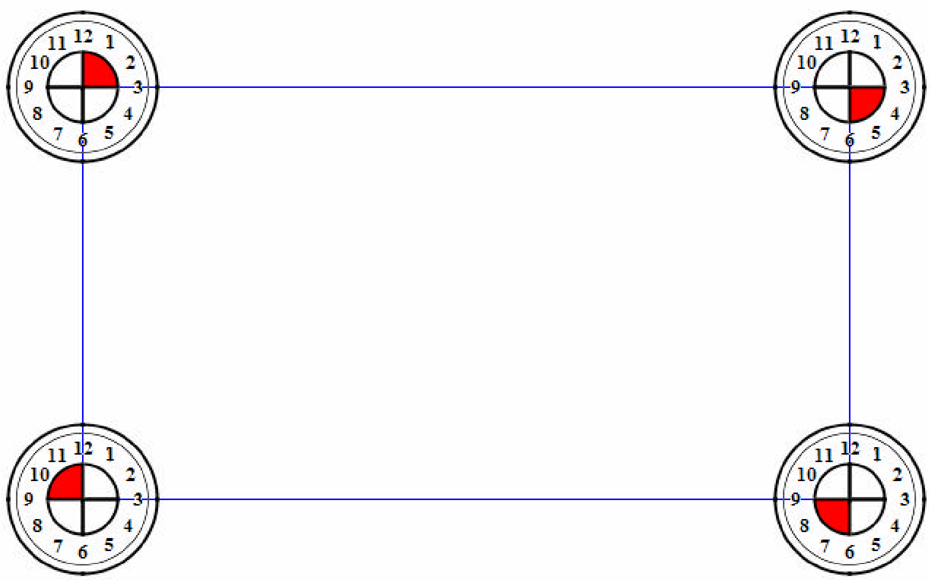

Figure 4. Walking the line with clocks is an early exercise to help the child associate number with change in direction and exterior angles of a figure. In the example above, the child begins walking from the lower left-hand corner of the rectangle to the upper left-hand corner. Upon reaching the corner, the child counts "1-2-3" as she pivots in the direction of the upper right-hand corner. Upon reaching the upper right hand corner, she counts "4-5-6" as she pivots toward the lower right-hand corner. The child continues until she returns to the lower right-hand corner and pivots to return to her original heading (facing the upper left-hand corner) while counting "10-11-12". The same exercise can be done using different figures whose exterior angles are multiples of 30, for example a regular triangle, a regul ar hexagon, or a 30-60-90 triangle. In all these cases, the child will find that the total amount of turning is always 12 "hours".

Next, we further help the child separate changes in position from changes in direction by suggesting that she count how much walking she has done "all together" by counting the total number of tiles covered (or steps taken) when walking the line (sides of a figure could also be marked off by making "tick marks" with tape so that children can measure distances on shapes other than rectangles).

Further extensions of walking the line and tracing shapes

After counting the total distance covered (the perimeter of the shape), the child can then count the total turn (the sum of the external angles). For any simple closed shape, the total turning will always be a complete circle or twelve hours as measured by the clocks on each corner.

This work should also be done with outlines of shapes and clocks drawn on each corner to give the bird's eye view. This activity extends the work with fraction circles and polygons described above, reinforcing the Total Turtle Trip Theorem, and further supporting connections between turtle turns and angles. An important point of interest here is that the total amount of turning done in traversing the figure (and returning to the original orientation) is the same amount of turning that would be done if the turtle stayed in one place and turned all the way around (twelve "hours" or 360 degrees).

Using the clock at each corner rather than a 360 degree protractor has the advantage that the child can quickly count through all of the turns. This aids the child in making the connection that one complete revolution is made in traversing the outline, and gives time to repeat the exercise on the same figure or different figures. It also helps reinforce the notion of right turns as clockwise turns and left turns as counterclockwise turns, which helps the child orient herself when taking the bird's eye view.

At some point, if the child does not spontaneously recognize the fact, it should be pointed out that "all the way around the circle" (and fractions of a circle) can be measured differently. With the clock, a complete revolution is twelve hours. With the centesimal frame, a complete revolution is 1.00. For the 360 degree protractor, a complete revolution is 360 degrees.

It would be nice to have a fraction circle divided into 12 pieces so we could easily relate "hours" and degrees.

Variations on playing turtle

Another exercise to help children develop skill in Turtle Geometry is done in pairs. One child has a picture of a geometric object with a point in the center of the drawing. The other child has a sheet with center marked, a 360 degree protractor and a ruler. At any point in the game, the child with the picture can give one of two commands to the other child—“right m” or “forward n”, where m is some number of degrees and n is some number of centimeters. This gives the child experience in tracing paths, and reinforces the idea that external angles are supplementary to internal angles. Kids should also “play Turtle” to trace different paths, one giving another commands. As with clock activity, kids need experience with both perspectives.

Work on the computer

There are a wide range of opinions about the appropriateness of computers for the Montessori classroom. Among those who feel that computers are appropriate, there is disagreement about the age at which computers should be introduced, how the computer should be presented, how the computer environment should be prepared, and how software and peripherals should be evaluated and used.

This section is for Montessori teachers who think the computer could be a good medium for helping their kids with geometry. In it, I present a software package called Circular Reasoning that I developed to provide links between Montessori materials (specifically, the fraction circles and the protractor used with fraction circles) and Turtle Geometry, and to link both to alternative representations of angles. Next, I will share a selection of web-based activities that I developed for classes I've taught in Turtle Geometry. The web-based activities are written in Logo and provide further links between Turtle Geometry, Montessori exercises in geometry, and Euclidean geometry.

Circular Reasoning is available for free from my web site, www.leonelearningsystems.com. It is inspired in part by the design work with geometric insets. Montessori wrote:

"The designing done with these geometric insets, as will be explained, is of two kinds: geometric and artistic (mechanical and decorative). And the union of the two kinds of drawings gives new ways of applying the material… [In geometric design, the child] acquires… actual and real cognitions in geometry… [The artistic design work] facilitates the development of the child's esthetic sense (Montessori, 1965)."

As mentioned earlier, the emphasis in the Montessori method is on "efficient experiences" rather than "construction in the world". However, she did feel that construction in the world could support important kinds of learning. The design work done with metal insets is one example of this:

"…The material of the geometric insets may be applied also to design. It is through design that the child may be led to ponder on the geometric figures which he has handled, taken out, combined in numerous ways, and replaced. In doing this he completes an exercise necessitating much use of the reasoning facilities. (Montessori, 1965)"

Circular Reasoning is a drawing environment that provides sectors (fractions of a circle) that children can manipulate in various ways (Figure 5).

Work on the computer (continued)

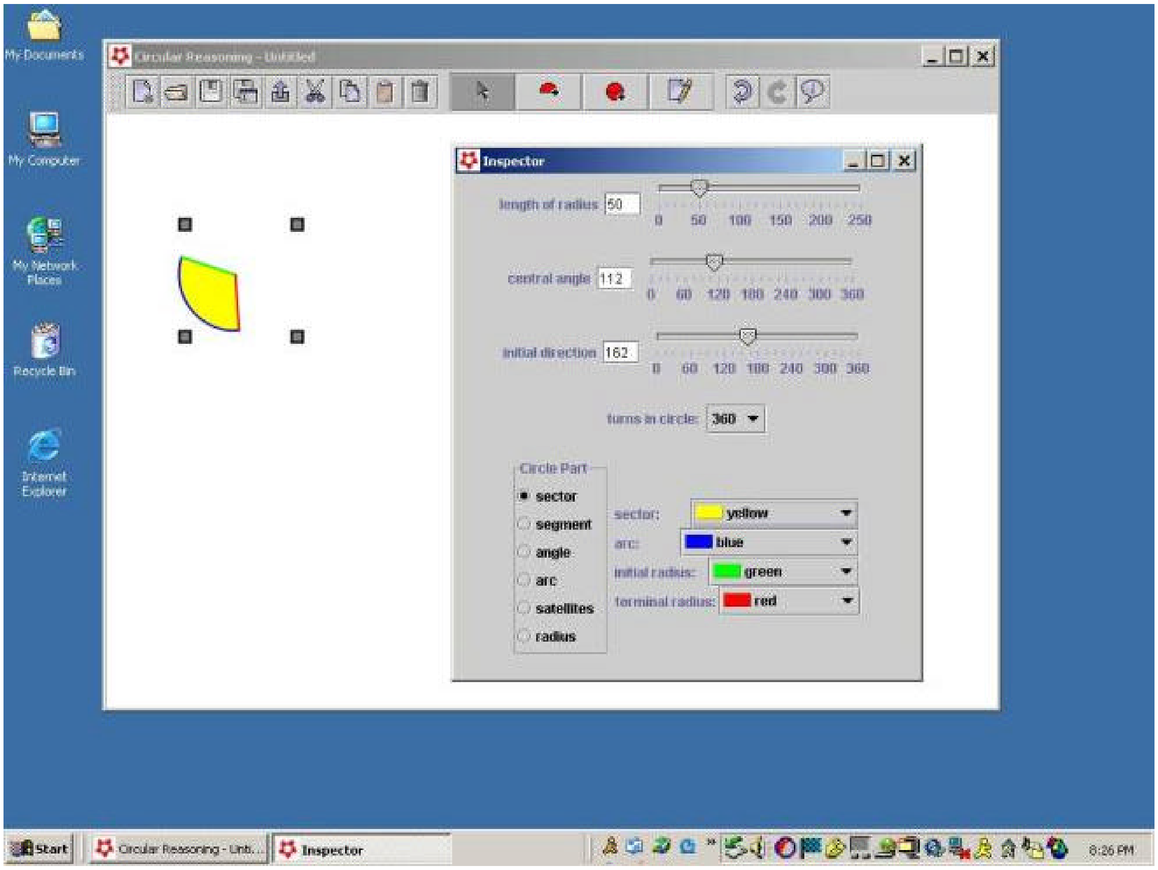

Figure 5. After a sector is placed in the workspace, the child can change its size, angle or direction, or change its representation to segment, angle, arc, satellites or radius. The various components of each representation can be separately colored.

The other primary screen objects in Circular Reasoning are the fraction circles and the free-floating notes. The fraction circles are circles divided into equal sectors. The number of sectors, representation of sectors, and relative position of sectors can be manipulated by the child. The notes are movable text boxes that children can use to make comments on their illustrations.

This software provides important links between Montessori activities and Logo. First, it uses screen objects (sectors and fraction circles) that are familiar to Montessori children. Next, it gives the student the opportunity to manipulate length of the sector radius, central angle of the sector, and orientation of sector separately by means of sliders that are marked off in pixels or degrees, as appropriate.

Work on the computer (continued)

As with traditional Montessori exercises in geometry, Circular Reasoning activities begin with perceptual level work with whole objects (sectors and fraction circles) and then move to analysis. Analysis is supported by supporting the child's ability to color (and so highlight) various parts of the sectors (radii, arc, area), to select different representations of angles (sectors, segments, two lines joined at a point, arc, orbit of satellite, inclination of a line), and to change spatial attributes (lengths of radii, central angle, and heading) with sliders and text boxes.

The ability to visually manipulate angles and side lengths with the slider has two important effects. First, it allows the student to visually and dynamically separate changes in side length from changes in angle. This helps with the development of the angle concept and with separation of change in position from change in direction. Second, it helps the child learn to estimate distances and angles, which is useful for work in Turtle Geometry. This second effect requires exercises that explicit ask the child to attend to the numbers on the various sliders.

I used this software package during fifteen hours of geometry classes which I taught for Northwestern University's Center for Talent Development. The course was called "Thinking About Circles." Classes were given over a period six weeks to a group of gifted 3rd and 4th grade children.

Children were given exercises to develop vocabulary (Figure 6) and recognize relationships between geometric objects (Figure 7).

Geometric Illustrations

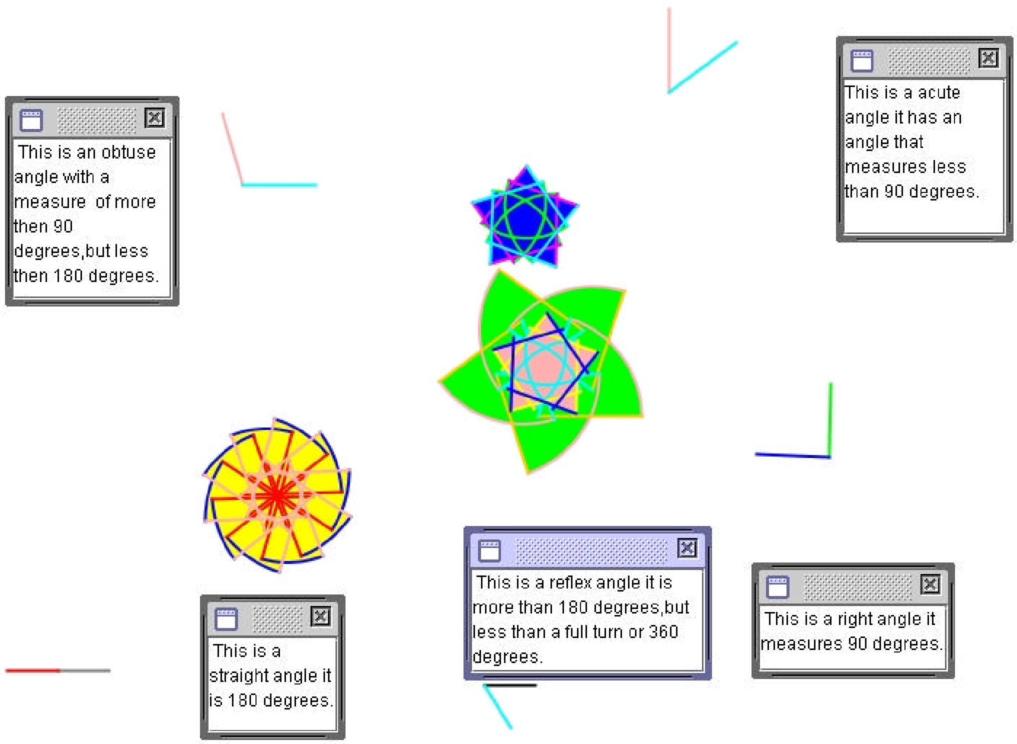

Figure 6. Sample of child's work. Task was to create illustrated definitions of various geometric objects.

Puzzles

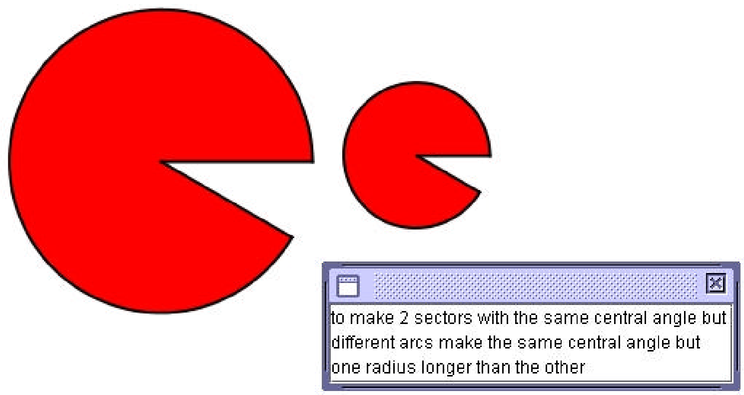

Figure 7. Sample of child's work. Puzzle posed to children was: Can you make two sectors with the same central angle but differently sized arcs?

Children also had ample opportunity to do representational (Figure 8) and abstract (Figure 9) art work with Circular Reasoning.

Representational Art

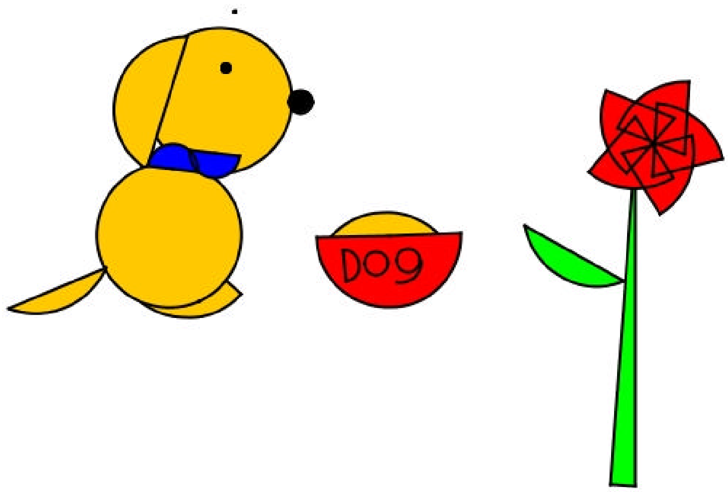

Figure 8. Sample of student's work. Flower and puppy made from sectors, segments, fraction circle and circle centers (points).

Abstract Art

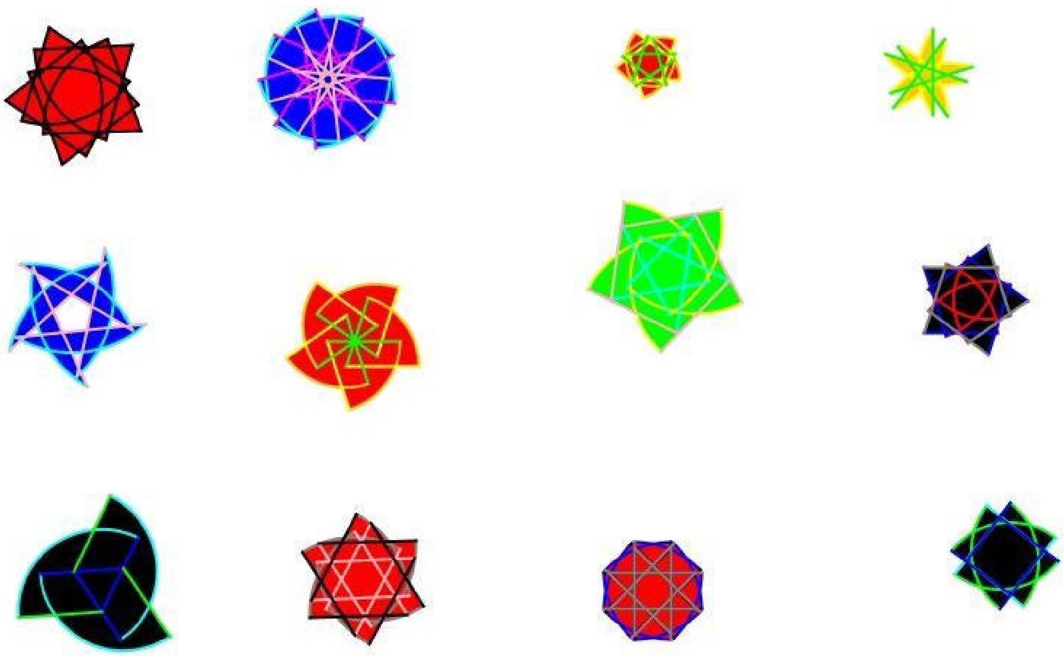

Figure 9. Sample of child's work. Geometric designs created by manipulating fraction circles.

Test Results

The Thinking About Circles class was given pre-tests and post-tests using selected questions in geometry (two open-ended and 7 multiple choice) from the National Assessment of Educational Progress (*NAEP Questions*, 2003) that were administered nationally to 8th and 12th graders.

On the pre-test, as we would expect, their performance was well below the performance of the 8th and 12th graders (*NAEP Questions*, 2003). The TAC kids had an average of 29% on the multiple-choice questions. Fourteen students took the pre-test. Seven of them got only one question right on the pre-test.

On the post-test, the TAC kids scored an average of 56% on the multiple choice questions, outperforming 8th graders on all questions administered to 8th graders except one, usually by a wide margin (Table 1).

Table 1. Comparative scores of Thinking About Circles class with national scores for 8th and 12th graders on selected NAEP geometry questions. Two scores are given for the Thinking About Circles class. "CR Pretest" are scores obtained before start of class. "CR Posttest" are scores after approximately twelve hours of discussion and work with Circular Reasoning software.

Comparative Scores

Test Question	3	4	5	6	7	8	9
CR Pretest							
Correct	31%	8%	15%	69%	31%	23%	31%
Incorrect	62%	54%	85%	23%	54%	54%	46%
Omitted Item	8%	38%	0%	8%	15%	31%	23%
8th Grade Scores							
Correct	32%	23%	33%	74%	-	-	-
Incorrect	67%	73%	66%	25%	-	-	-
Omitted Item	10%	4%	1%	1%	-	-	-
12th Grade Scores							
Correct	-	44%	-	-	70%	49%	79%
Incorrect	-	53%	-	-	29%	49%	20%
Omitted Item	-	3%	-	-	1%	2%	1%
CR Posttest							
Correct	46%	46%	31%	92%	46%	54%	77%
Incorrect	54%	23%	69%	8%	36%	46%	23%
Omitted Item	0%	31%	0%	0%	15%	0%	0%

Four of the questions on the test were administered nationally to 12th graders. Of those four questions, the TAC kids outperformed 12th graders on two of the questions (*NAEP Questions*, 2003).

Test Results (continued)

The TAC class had one 4th grader and all the rest were 3rd graders and about 12 hours between tests for work with Circular Reasoning and classroom discussion (15 hours in attendance minus time for snacks and tests).

Student motivation generally appeared high, though no attempt was made to apply specific measures of motivation. Further study of Circular Reasoning activities is required to corroborate current findings on a wide scale and examine other dimensions of their effects, including before and after assessments of van Hiele levels of reasoning, how or whether students achieve flow states, boredom, or anxiety while engaged in Circular Reasoning activities, and longitudinal effects.

Logo Activities

I have also conducted classes on Turtle Geometry using web activities that I developed with MicroWorlds Logo. These exercises give further opportunities for children to develop connections between turtle turns and static representations of angles, such as fractions (sectors) from a fraction circle. I will present three of these exercises below. The exercises can be viewed and used from my web site (www.leonelearningsystems.com).

Installation of the MicroWorlds Web Player is required (http://microworlds.com/webplayer/index.html). Unfortunately, the MicroWorlds Web Player is only officially supported under Internet Explorer and Netscape 4.0 or higher. Also, as of this writing, there is not yet a MicroWorlds Web Player under MacOS X for projects made with MicroWorlds EX, so some of my site's pages with MicroWorlds projects won't run on a Macintosh.

The first of these exercises is a game of Simon Says with a turtle (Figure 10). This exercise gives the child experience in observing right turns of 0, 90, 180, and 270 degrees from different starting orientations.

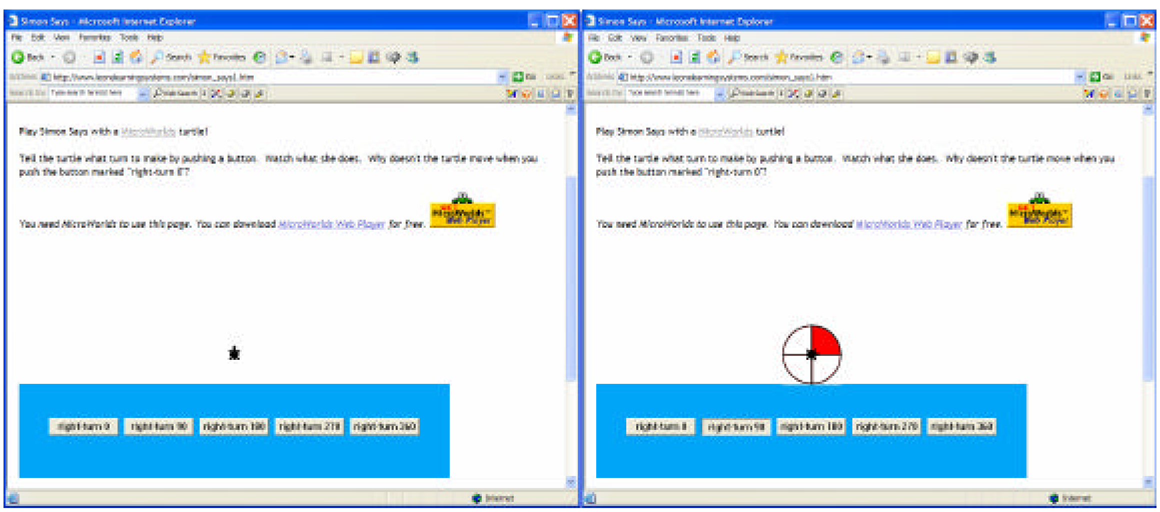

Figure 10. In this exercise, the child experiments with different turtle turns. The web page initially appears as the panel on the left. When a button is pressed (in this case, the right-90 button), a circle appears that is divided into four sectors. The sectors that correspond to the turtle turn appear in red, and the turtle turns to its new heading, as shown in the second panel. After the turtle has completed the turn, the sectors disappear. Sectors and turns are always made relative to the heading of the turtle at the time the button is pressed.

Logo Activities (continued)

The following exercise corresponds to the work with polygons and fraction circles described earlier (Figure 11). In addition to using red sectors for external angles, explicit representations of internal angles are also provided in the form of blue sectors. This activity can be used later to show an interesting derivation of the formula for the sum of the internal angles of a figure.

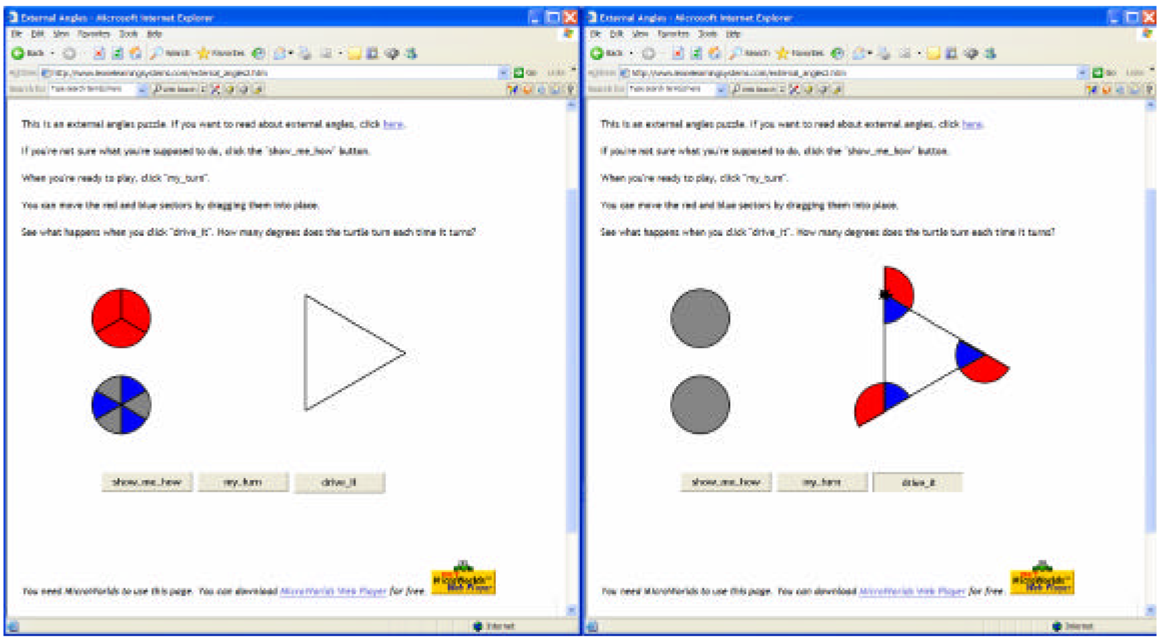

Figure 11. In this exercise, the child relates external angles to turtle turns made while traversing the path of an equilateral triangle. In the first part of the exercise, the child moves the red sectors into positions corresponding to external angles, and moves blue circles into positions corresponding to internal angles. The "show me how" button demonstrates this activity for children who need a demonstration. After the sectors are in place, the child can press the "drive it" button to see the turtle "drive" around the outline of the triangle. Each of its turns correspond to one of the red sectors.

We can see from the figure that each red and blue sector form a pair that make half a circle. There are three such pairs, so we have three circle halves in all. We know that the red sectors add up to a full circle, or two halves of a circle. So the blue (internal) angles must add up to half a circle. Once the child has internalized the Total Turtle Trip Theorem (turns around a simple closed figure add up to 360 degrees), we can use the fact that internal and external angles are supplementary to generalize that for any *n*-gon (whether or not it is regular), the sum of the external angles is 360, and the sum of the internal angles is $180n - 360$.

Logo Activities (continued)

The next exercise supports the child's understanding of turtle turns as supplementary to the angle drawn by the turtle (Figure 12). It also gives support for learning to measure with a 180 degree protractor.

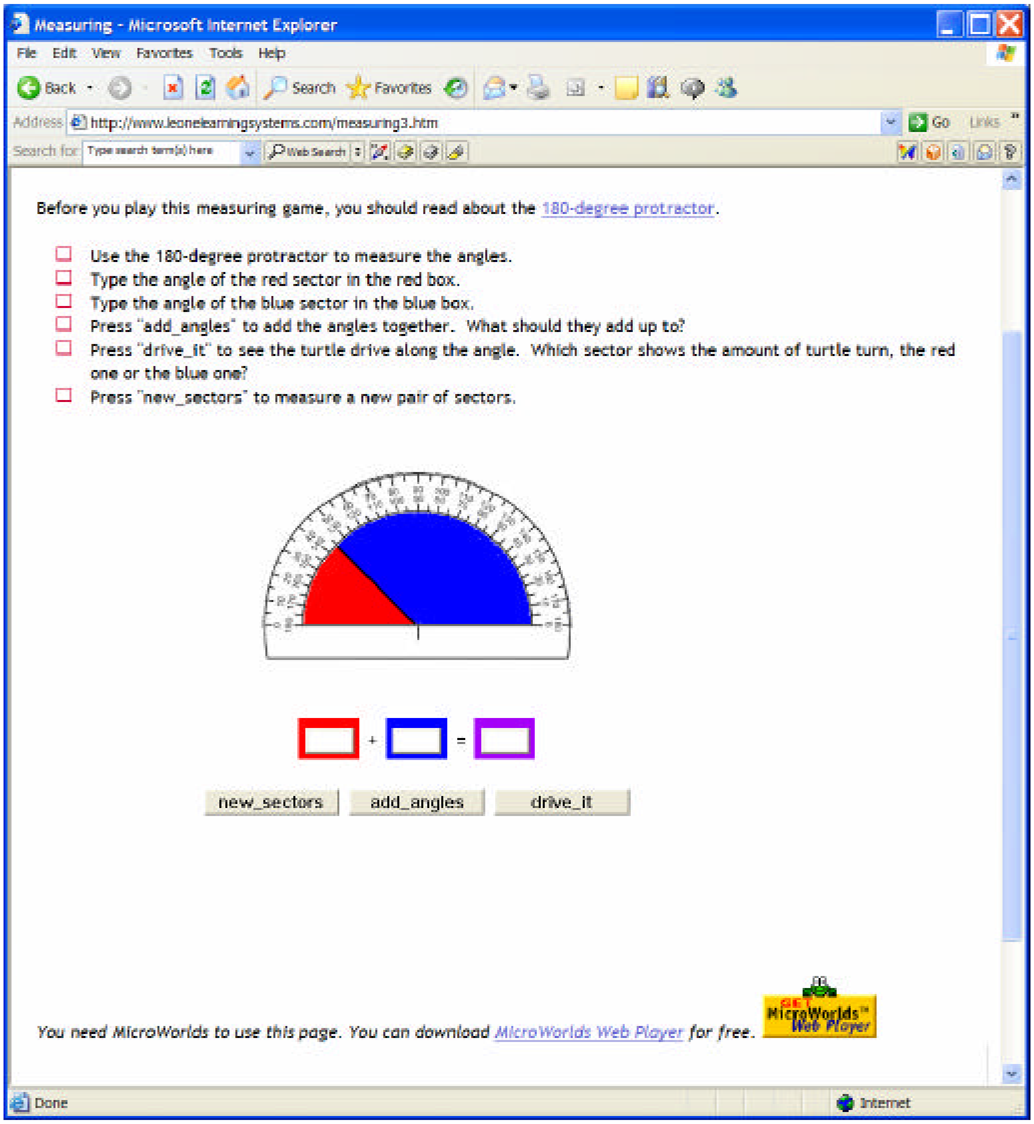

Figure 12. In this exercise, the child uses sectors and turtle turns in work with a 180 degree protractor. The degree measure of the red sector is entered in the red box. The degree measure of the blue sector is entered in the blue box. When the child clicks the "add angles" button, the sum of the two angles appears in the purple box. At this point, the child presses the "drive it" button to see the turtle drive from the right of the protractor to the center, make a right turn that corresponds to the angle entered in the red box, and move forward a distance that corresponds to the radius of the semicircle. This provides feedback that tells the child if the correct angle was entered in the red box. Other kinds of feedback (for example, if the child did not enter any numbers in the boxes) is given through dialog boxes.

Logo Activities (continued)

To continue exploration of Turtle Geometry with Logo, the child needs familiarity of the Logo programming language, especially the concepts of command, operation, value passing, variable, and iteration.

Other Areas for Discussion Between Montessorians and Constructionists

There are a number of aspects of children's work that were not explicitly addressed in this article, where dialog between constructionists and Montessorians could also prove fruitful. These include the roles of emotions and imagination in learning, supporting group work, supporting open-ended work, and classroom management in constructivist environments.

Further Reading

Seymour Papert's first book on Logo is *Mindstorms: Children, Computers, and Powerful Ideas*. His second book, The Children's Machine: Rethinking School in the Age of the Computer, includes a chapter called Instructionism versus Constructionism, which compare constructionism to direct instruction.

Since the Logo language itself is beyond the scope of this article, I will direct the reader to Brian Harvey's three volumes titled *Computer Science Logo Style*, especially the first volume called *Symbolic Computing* (Harvey, 1997), which includes a wonderful off-line activity on variables (see "The Little Person Metaphor" in chapter 3) which can also be used to help the child understand value passing from operations to other operations and commands (in the MicroWorlds version of Logo, operations are referred to as *reporters*).

I also deal with these concepts in a text I wrote to introduce children to algebra concepts with Logo. The Brian Harvey texts and mine are both available for free on the web. Volume 1 of Brian Harvey's work is at http://www.cs.berkeley.edu/~bh/v1-toc2.html. My text, *Beginning Algebra*, is at http://www.leonelearningsystems.com/beginning_algebra.htm. For a much deeper text on exploring algebra with Logo, see *Investigations in Algebra* by Albert Cuoco (Cuoco, 1990).

For much deeper treatments of Turtle Geometry as implemented in Logo, with links to Cartesian geometry, vector analysis, and more, I recommend *Turtle Geometry* by Harold Abelson and Andrea diSessa (Abelson & diSessa, 1986), and *Approaching Precalculus Mathematics Discretely*, by Philip G. Lewis (Lewis, 1990).

For a great overview of research on children's learning with Logo, see *Research on Logo: Effects and Efficacy* by Douglas H. Clements and Julie S. Meredith (Clements & Meredith, 1992). Douglas H. Clements also produced a monograph for the National Council of Teachers of Mathematics (NCTM) with Michael T. Battista and Julie Sarama, called *Logo and Geometry*.

The monograph reports on studies of the Logo Geometry Project, a "research-based Grades K-6 geometry curriculum" (Clements, Battista, & Sarama, 2001) which was developed over a four year period. The book Learning Mathematics and Logo by Celia Hoyles and Richard Noss provides an excellent selection of essays on research and curriculum development using Logo as a medium for explorations in mathematics.

Although it was not explicitly discussed in this article, the van Hiele model of children's thinking in geometry was an important part of the theoretical underpinning of my work. Like Montessori and Piaget, the van Hieles conceived of child development in stages. The focus of their work was the child's developing understanding of geometric concepts.

Further Reading (continued)

I recommend The van Hiele Model of Thinking in Geometry Among Adolescents (Fuys, Geddes, & Tischler, 1988) for those interested in looking a research and curriculum project based on the van Hiele model.

Another important influence on my work is Hans Freudenthal. In particular, his analysis of multiple angle concepts (Freudenthal, 1973) sheds light on problems that children have in understanding the turtle's perspective while viewing the turtle on the screen, and provides theoretical support for my use of multiple representations of angles in Circular Reasoning©.

Former students of Papert have written books that further elaborate on his theory and report on classroom experiences with constructionism. These researchers include Idit Harel, Yasmin Kafai, and Mitchel Resnick. For a discussion of constructionist activities that are not computer-based, see the paper *Design of an Environment for Learning about Topology and Learning about Learning*, by Carol Strohecker (Strohecker, 1996).

Constructionist Resources

In theory, constructionism does not require the use of a computer. In practice, however, most of the original work in constructionism involved activities with Logo, and much of it still does. There are a large number of Logo versions available, and many of them are free. An extensive list of different versions can be found at http://www.logosurvey.co.uk/. This site also has a lot of other information on Logo. Other good starting points for learning about Logo include The Logo Foundation (http://el.media.mit.edu/logo-foundation/) and the MicroWorlds (http://www.microworlds.com/) web sites.

Of the free versions, I recommend Brian Harvey's Berkeley Logo (http://www.cs.berkeley.edu/~bh/), George Mills' MSWLogo (http://www.softronix.com/logo.html), and NetLogo (http://ccl.northwestern.edu/netlogo/). MSWLogo provides a more Windows-compatible GUI to Berkeley Logo, but it only runs in Windows. Berkeley Logo runs on a wide range of machines, including Windows PCs and macs. NetLogo supports simultaneous work with huge numbers of turtles.

There are also a number of commercial versions of Logo available. One that I have personally used in classrooms is MicroWorlds Logo (http://www.microworlds.com/). It has a lot of bells and whistles that may distract kids, depending on what learning you're trying to support. When I want to use Logo specifically for work in math or list processing, I give the kids Berkeley Logo or MSWLogo. However, MicroWorlds is great for game development and multimedia projects. I also find it useful for creating web-based Turtle Geometry exercises.

For examples of computer-based constructionist exercises don't use Logo, see www.MaMaMedia.com.

The Resources page of the Logo Foundation web site (http://el.media.mit.edu/logo-foundation/resources/index.html) provides more information about Logo-related web sites, Logo users groups, and Logo online discussion groups. Discussion groups are a great place for Montessorians to initiate discussions with constructionists.

I would also love to hear from other Montessorians who are using Logo or other constructionist materials (including, but not limited to, software and robotics) in their classrooms. I'm curious to know what benefits you may have found from using these materials and how you apply Montessori principles to their use. Conversely, I'm also interested in how constructionist principles might inform any of the activities in your Montessori classroom.

Please also contact me if you have any questions or comments about the exercises described in this article. My e-mail address is tj@leonelearningsystems.com.

References

Abelson, H., & diSessa, A. (1986). *Turtle Geometry: The Computer as a Medium for Exploring Mathematics* (First MIT Press paperback edition ed.). Cambridge, MA
London, England: The MIT Press.

Chattin-McNichols, J. (1992). *The Montessori controversy*. Albany, NY: Delmar.

Clements, D. H., Battista, M. T., & Sarama, J. (2001). *Logo and Geometry* (Reports - Research No. SE066038). Reston, VA: National Council of Teachers of Mathematics, Inc.

Clements, D. H., & Meredith, J. s. (1992, 2000). *Research on Logo: Effects and Efficacy*. Retrieved January 2, 2004, 2004, from http://el.media.mit.edu/logo-foundation/pubs/papers/research_logo.html

Cuoco, A. (1990). *Investigations in algebra*: MIT Press.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht, Holland: D. Reidel Publishing Company.

Fuys, D., Geddes, D., & Tischler, R. (1988). *The van Hiele Model of Thinking in Geometry among Adolescents* (Reports - Descriptive; Reports - Research No. SE049183). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Gebhardt-Seele Peter, G. (1985). *The computer and the child: a Montessori approach*. Rockville, MD.: Computer Science Press.

Harvey, B. (1997). *Symbolic Computing* (2nd ed. Vol. 1). Cambridge, MA: The MIT Press.

Hoyles, C., & Noss, R. (1992). *Learning mathematics and Logo*. Cambridge, MA; London: MIT Press.

Lewis, P. G. (1990). *Approaching Precalculus Mathematics Discretely: Explorations in a Computer Environment*. Cambridge, MA
London, England: The MIT Press.

Loeffler, M. H. (1992). Montessori and Constructivism. In M. H. Loeffler (Ed.), *Montessori in contemporary American culture* (pp. 101-113). Portsmouth, NH: Heinemann.

References (continued)

Montessori, M. (1964). *The Montessori method*. New York,: Schocken Books.

Montessori, M. (1965). *The Advanced Montessori Method II: Scientific Pedagogy as Applied to the Education of Children from Seven to Eleven Years* (A. Livingston, Trans. Vol. 13): Clio Press.

NAEP Questions. (2003, September 9, 2003). Retrieved April 10, 2003, from http://nces.ed.gov/nationsreportcard/itmrls/

Papert, S. (1993a). *The children's machine : rethinking school in the age of the computer*. New York: BasicBooks.

Papert, S. (1993b). *Mindstorms: children, computers and powerful ideas* (2nd ed ed.). New York ; London: Harvester Wheatsheaf.

Prescott, A., Mitchelmore, M., & White, P. (2002, July, 2002). *Student Difficulties in Abstracting Angle Concepts from Physical Activities with Concrete Materials.* Paper presented at the The 25th Annual Conference of the Mathematics Education Research Group of Australia Incorporated, Auckland, New Zealand.

Strohecker, C. (1996). *Design of an Environment for Learning about Topology and Learning about Learning*. Retrieved February 3, 2005, from http://www.merl.com/papers/docs/TR96-05.pdf

What is Logo? (2000). Retrieved January 25, 2004, from http://el.media.mit.edu/logo-foundation/about/index.html

The Author

TJ Leone develops and implements computer-based and offline enrichment activities and software for children in after-school and distance education programs, and in individual sessions. He has a bachelor's degree in mathematics, an MS in computer science, and further graduate work in Learning Sciences. He spent six years developing educational software for middle and high school students at Northwestern University's School of Education and Social Policy, and three years working with master AMS and AMI teachers in Montessori classrooms for children aged 3 to 12. His web site is http://www.leonelearningsystems.com.